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General relativistic gravitational field effects on superfluid phase interference devices

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Abstract. In some recent experiments, it has been established that the Newtonian gravitational potential can influence the phase interference between alternative virtual paths taken by a quantum test particle. Here, we consider theoretically the influence of general relativistic fields on the phase interference of superfluid flows.

1. Introduction

In the Feynman path integral formulation of quantum mechanics, the amplitude for a test particle to travel along a path P is related to the action for the path by

$$\mathrm{Amp}(P) = \exp(\mathrm{i}/\hbar)S(P). \tag{1}$$

Alternatively, the phase interference in the amplitudes for two paths of a quantum test particle, which have the same initial and final points, is given by

$$\hbar\theta(P_1, P_2) = S(P_1) - S(P_2). \tag{2}$$

Recent experimental evidence (Werner *et al* 1975) has been obtained that the Newtonian gravitational potential can contribute to θ for a non-relativistic quantum test particle.

Equation (2) will have a particularly interesting form when the physical external field acting on the test particle is derived from a gauge field, such as the electromagnetic field and the general relativistic gravitational field. Here, the phase interference function is (in principle) a measurable quantity, and will not depend on the particular choice of gauge. This becomes possible because the physical external field acts on the quantum test particle from a distance which yields a generalised Bohm–Aharonov (Aharonov and Bohm 1959) effect.

In a superfluid system, the phase interference θ can appear on the macroscopic scale of superfluid flows. The gauge field contributions to θ will remain intact, if the superfluid order parameter transforms as a single test particle wavefunction under changes in the gauge fields. For example, the electromagnetic flux contribution to the phase of the Josephson current (Feynman 1965) in a superconducting quantum interference device appears because the Landau–Ginzburg order parameter transforms (under electromagnetic gauge change) as a spinless boson wavefunction of charge $q = 2e$.

The purpose of this work is to discuss the influence of the general relativistic gravitational field on the superfluid phase. As an explicit example, we choose to compute the influence of neighbouring rotating masses on the phase interference in a quantum superfluid device due to general relativistic gravitational effects. Although the engineering problems in detecting general relativistic effects via the superfluid phase are formidable, the theoretical predictions are clear and worthy of note.

2. The electromagnetic field

For the purpose of comparison, let us first briefly review the physical principles by which electromagnetic flux is detected in a superconducting quantum interference device (SQUID). The gauge field is given by the differential form

$$A = \sum_{\mu} A_{\mu}\, \mathrm{d}x^{\mu}, \tag{3}$$

which yields a contribution to the path action of a quantum test particle of charge q,

$$\Delta S(P) = (q/c) \int_P A. \tag{4}$$

The phase interference implied by equations (2) and (4) is given by

$$\Delta\theta = (q/\hbar c)\left(\int_{P_1} - \int_{P_2} \right) A = (q/\hbar c) \oint A. \tag{5}$$

In terms of the physical electromagnetic field differential form

$$F = \mathrm{d}A = \tfrac{1}{2} \sum_{\mu\nu} F_{\mu\nu} (\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}), \tag{6}$$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \tag{7}$$

the integral on the right-hand side of equation (5) can be written as the electromagnetic flux through a surface bounded by the closed path,

$$\Phi = \oint A = \iint F. \tag{8}$$

Hence

$$\Delta\theta = (q\Phi/\hbar c), \tag{9}$$

which is the Bohm–Aharonov effect, from a relativistic viewpoint.

In a superconducting ring with a Josephson weak link, the phase interference is from clockwise and counter-clockwise electron pair paths (Widom and Clark 1980) ($q = 2e$) so that the total ring current (summed for both orientations) is

$$I = (I_0/2\mathrm{i})(\mathrm{e}^{\mathrm{i}\Delta\theta} - \mathrm{e}^{-\mathrm{i}\Delta\theta}), \tag{10}$$

which is the Josephson law relating current and electromagnetic flux; i.e.

$$I = I_0 \sin(2\pi\Phi/\Phi_0), \tag{11}$$

where $\Phi_0 = (\pi\hbar c/e)$ is the flux quantum.

Let us now consider the analogous arguments for the general relativistic gravitational field.

3. Gravitational fields

The path action for a test particle of mass m moving in a gravitational field (Landau and Lifshitz 1975a) is given by

$$S(P) = -mc \int_P \mathrm{d}\sigma, \tag{12}$$

where the interval is given by the space–time metric

$$-\mathrm{d}\sigma^2 = \sum_{\mu\nu} g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu. \tag{13}$$

In terms of the path velocity

$$v^\mu = c(\mathrm{d}x^\mu/\mathrm{d}\sigma), \tag{14}$$

the action is determined by

$$S(P) = m \int_P v, \tag{15}$$

$$v = \sum_\mu v_\mu\, \mathrm{d}x^\mu. \tag{16}$$

Equations (2) and (15) imply a phase interference function

$$\theta = (m/\hbar)\left(\int_{P_1} - \int_{P_2}\right) v = (m/\hbar) \oint v. \tag{17}$$

The role of the flux quantum (determined by the test particle charge)

$$\Phi_0 = (2\pi\hbar c/q), \tag{18}$$

in the electrodynamic case, is here played by the Compton wavelength (determined by the test particle mass),

$$\lambda_0 = (2\pi\hbar/mc), \tag{19}$$

i.e. the phase interference function reads

$$\theta = (2\pi/\lambda_0) \oint (v/c). \tag{20}$$

Contributions to the phase interference function in equation (20) can arise even when the gravitational field vanishes. Hence, the gravitational field contribution to equation (20) must be extracted with care. As an example of phase interference in flat space–time, consider a superfluid constrained inside a ring of radius R rotating about the ring axis of symmetry with angular velocity Ω. In a cylindrical rotating frame coordinate system the flat space–time metric reads

$$\mathrm{d}\sigma^2 = (c^2 - \Omega^2\rho^2)\, \mathrm{d}t^2 - (\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\, \mathrm{d}\varphi^2) - 2\Omega\rho^2\, \mathrm{d}\varphi\, \mathrm{d}t. \tag{21}$$

Presuming that the test particle velocity $|\boldsymbol{v}| \ll c$ and the tangential ring (rotational) velocity $(\Omega \times R) \ll c$, the phase interference between clockwise and counter-clockwise paths implied by equations (20) and (21) is the well known rotating superfluid quantum interference device result

$$\Delta\theta = (2\pi/\lambda_0)(2\pi R^2\Omega/c). \tag{22}$$

For a charged superfluid, rotational velocities have an equivalent magnetic field strength ($q = 2e$),

$$\boldsymbol{\Omega} = (q/2mc)\boldsymbol{B}_{\text{eff}}, \tag{23}$$

from the viewpoint of phase interference.

Now let us consider a case which is an actual general relativistic gravitational field effect. Consider a sphere with a large mass M, centre at the origin, and having a rotational angular momentum $\boldsymbol{L}$ about an axis through the sphere centre. Let the superfluid quantum interference device be placed in a spatial neighbourhood of the rotating sphere. The lowest-order deviations from flat space–time due to the rotating sphere in the region of the phase detecting device can be written as (Landau and Lifshitz 1975b)

$$d\sigma^2 = (1 + 2\varphi/c^2 + \ldots)(c\,dt - \boldsymbol{a} \cdot d\boldsymbol{r} + \ldots)^2 - |d\boldsymbol{r}|^2 + \ldots, \tag{24}$$

where

$$\varphi = (GM/r) \tag{25}$$

is the Newtonian potential,

$$\boldsymbol{a} = (2G/c^3r^3)\boldsymbol{r} \wedge \boldsymbol{L}. \tag{26}$$

The Coriolis vector

$$\boldsymbol{\Omega} = (c/2)\,\text{curl}\,\boldsymbol{a} \tag{27}$$

can be detected as in equation (23). Specifically, the phase interference in a closed superfluid detector path due to the neighbouring rotation in the sphere is given by

$$\Delta\theta = (4\pi/\lambda_0)(G/c^3)\boldsymbol{L} \cdot \oint \left(\frac{d\boldsymbol{r} \wedge \boldsymbol{r}}{r^3}\right), \tag{28}$$

where λ_0 is the Compton wavelength of the detector test particle (e.g. an electron pair). In terms of the Planck length Λ, where

$$\Lambda^2 = (\hbar G/c^3), \tag{29}$$

and the macroscopic angular momentum in quantum units ($\boldsymbol{L}/\hbar$), equation (28) reads

$$\Delta\theta = (4\pi\Lambda^2/\lambda_0)(\boldsymbol{L}/\hbar) \cdot \oint (d\boldsymbol{r} \wedge \boldsymbol{r}/r^3). \tag{30}$$

In a superconducting quantum interference device magnetometer shielded (by other superconductors) from magnetic fields, the neighbouring rotating mass will produce (in the detector) a Josephson current

$$I = I_0 \sin(\Delta\theta), \tag{31}$$

where $\Delta\theta$ is given in equation (30).

4. Conclusion

For a general metric, the phase interference function can be computed from the general relativistic version of the London (1960) fluxoid

$$\hbar\theta = \oint \sum_{\mu} [mv_{\mu} + (q/c)A_{\mu}]\, dx^{\mu}. \tag{32}$$

This evidently has the form

$$\theta = 2\pi[(l/\lambda_0) + (\Phi/\Phi_0)], \tag{33}$$

where Φ is the electromagnetic field flux contribution, and the lengthscale l,

$$cl = \oint \Big(\sum_{\mu} v_{\mu}\, dx^{\mu} \Big), \tag{34}$$

will be influenced by space–time geometry even for the $\Phi = 0$ case.

In the voltage-current characteristics of a superconducting quantum interference device, the Faraday law voltage $V = -\dot{\Phi}/c$ depends only on the electromagnetic flux contribution to equation (33), while the Josephson current $I = I_0 \sin\theta$ depends on the total phase interference function.

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